Homework assignment 4

Due Weds, Feb 20th 2019.

Problem 1: Gaussian wave function

We have shown in class that function $f(x)$:

$$f(x) = \begin{cases} \frac{1}{a} & -a/2 \leq x \leq a/2 \\ 0 & \text{all other } x \end{cases} \tag{1}$$

is a δ-function when $a \to 0$.

(a) A Gaussian wave function can be written as $\psi(x) = Ne^{-x^2/a^2}$, where N and a are real constants. What is the value of N if $\psi(x)$ is properly normalized?

(b) We know that all wave function can be decomposed to the superposition of plane wave functions such that: $\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{\psi(k)} e^{ikx} dk$. Calculate $\widetilde{\psi(k)}$ for the Gaussian wave function in problem (a). (Hink: for the sake of your mental health, please use Mathematica if you can't figure out how to compute the integration. It's perfectly fine.)

(c) In classical physics, when solving the motion of a particle (or a wave), we need two initial conditions: (i) the position of the particle at time $t = 0$, and (ii) the speed (or momentum) of the particle at time $t = 0$. And these two conditions are independent. Knowing one of the condition does not give you any information about the other condition. Based on your answer in (b), do you think in quantum mechanics we need two initial conditions? Can you derive the momentum initial condition from the position initial condition? Do you think your answer makes sense?

(b) At $a \to 0$, is $\psi(x)$ a δ-function? If yes, prove that it satisfies the definition of δ-function.

Problem 2: More on Delta function

The heaviside function $H(x)$, or commonly known as step function, is defined as:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{2}$$

(a) What is the derivative of of $H(x)$ at $x \neq 0$?

(b) Is the derivative of $H(x)$ at $x = 0$ a finite number?

(c) Calculate $\int_{-\infty}^{\infty} \frac{\partial H(x)}{\partial x} dx$.

(d) Is $\frac{\partial H(x)}{\partial x}$ a δ-function?

(e) Use integration by part to show that $\int_{-\infty}^{\infty} f(x) \frac{\partial H(x)}{\partial x} dx = f(0)$. $f(x)$ is an arbitary function, and $f(x) \to 0$ at $x \to \pm\infty$.

(f) Do you think, in general, we can have $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$? Make an intuitive argument about your answer.

Problem 3: Properties of Delta function

Suppose $D_1(x)$ and $D_2(x)$ are two functions associated with delta-function. Mathematically, we say $D_1(x)$ equals to $D_2(x)$ if $\int_{-\infty}^{\infty} f(x)D_1(x)dx = \int_{-\infty}^{\infty} f(x)D_2(x)dx$ for any smooth $f(x)$. This is an interesting definition because it doesn't require $D_1(x) = D_2(x)$ at any given x.

Show that:

(a) $x\delta(x) = 0$.

(b) $\delta(-x) = \delta(x)$.

(c) $\delta(ax) = \delta(-ax) = \frac{1}{|a|}\delta(x)$.

(d) $\int_{-\infty}^{\infty} \delta(a-x)\delta(x-b)dx = \delta(a-b)$.

(f) Bonus problem: $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk = \delta(x)$. (Hint: Use Fourier transform and inverse Fourier transform to show that $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \int_{-\infty}^{\infty} f(y)e^{-iky}dy$. Compare it with $f(x) = \int_{-\infty}^{\infty} f(y)\delta(x-y)dy$.)

Problem 4: Operator

(a) Is e^{ikx} an eigen-function of momentum operator? If yes, what is the eigenvalue?

(b) Is $e^{ik_1x} + e^{ik_2x}$ an eigen-function of momentum operator? If yes, what is the eigenvalue?

(c) Is e^{ikx} an eigen-function of operator $\hat{H} = \hat{p}^2/2m$, the Hamiltonian operator of a free particle? If yes, what is the eigenvalue?

(d) Calculate $\hat{x}\hat{p}e^{ikx}$, and $\hat{p}\hat{x}e^{ikx}$. Are they the same?

(e) For the Gaussian function in problem 1 (a), what is its expected value of position x?

(f) What is its expected value of x^2?

(g) What is its expected value of momentum p?

(h) What is its expected value of momentum p^2?

(i) The uncertainty of a quantity is defined as its standard deviation

$$\Delta x = \sqrt{\frac{\sum_{m=1}^{N}(x_m - \bar{x})^2}{N}} = \sqrt{\overline{x^2} - \bar{x}^2}$$

, where the bar on top of the letter denotes expected value (or average value) of the quantity. Use the above answers from (e) to (h), calculate the uncertainty of x.

(j) Use the above answers from (e) to (h), calculate the uncertainty of p.

(k) What is the value of $\Delta x \Delta p$? Does it satisfy the uncertainty principle?