Homework assignment 5

Due Weds, Feb 27th 2019.

Problem 1: Property of eigenfunctions

We know that the eigenstates of a one-dimensional Hamiltonian satisfy the time-independent Schrodinger equation:

$$(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x))\psi(x) = E\psi(x). \tag{1}$$

(a) Suppose $\psi(x)$ is a solution to the time-independent Schrodinger equation. Is its conjugate, $\psi^*(x)$, also a solution to the time-independent Schrodinger equation?

(b) Is $\psi_R(x) = \psi(x) + \psi^*(x)$ a solution to the time-independent Schrodinger equation?

(c) Is $\psi_R(x)$ a real function, or a complex function?

(d) Do you think we can always choose eigenstates of the time-independent Schrodinger equation to be real?

(e) If the potential $V(x)$ is an even function (e.g. $V(x) = V(-x)$), and $\phi(x)$ is a solution to the time-independent Schrodinger equation. Show that $\phi(-x)$ is also a solution to the time-independent Schrodinger equation.

(f) Are $\phi_+(x) = \phi(x) + \phi(-x)$ and $\phi_-(x) = \phi(x) - \phi(-x)$ solutions to the time-independent Schrodinger equation?

(g) Are $\phi_+(x)$ and $\phi_-(x)$ even, or odd function? (definition: even function: $f(x) = f(-x)$. Odd function: $f(x) = -f(-x)$.)

(h) Do you think we can conclude that: when the potential $V(x)$ is an even function, we can always choose an even or odd function as the eigenfunction of the time-indepdent Schrodinger equation?

Problem 2: Infinite potential well

Consider a particle of mass m that is in a superposition state of the first two eigenstates of an infinite potential well of width L,

$$\psi(x,t) = \alpha \sin(\frac{\pi x}{L})e^{-i\omega_1 t} + \alpha \sin(\frac{\pi x}{L})e^{-i\omega_2 t}. \tag{2}$$

(a) According to your answers in Homework assignment 3, problem 2(d), can you write down ω_1 and ω_2 in terms of $\hbar$, m, and L?

(b) Normalization: we know that the probability of finding a particle in the infinite potential well is always 1. Can you use this condition to calculate α for $t = 0$?

(c) Use the α you derived from (b), now calculate the probability of finding a particle in the infinite potential well at arbitrary time t. Is the probability 1 for $t \neq 0$?

(d) What is the probability density of finding particle at position $x = L/2$ at arbitrary time t? Does it change with time?

(e) What is the probability density of finding particle at position $x = L/4$ at arbitrary time t? Does it change with time?

(f) What is the probability of finding the particle on the left side of the potential well ($0 < x < L/2$)? Does the probability change with time?

(g) What is the probability of finding the particle on the right side of the potential well ($L/2 < x < L$)? Does the probability change with time?

(h) Is the particle moving in the potential well? Is this a periodic movement?

(i) If you want to measure the energy of the particle in the potential well, what are the possible measurement results? What is the probability of getting each measurement result?

(j) Suppose you measure the energy, and you get energy $\hbar\omega_1$. What is the wave function of the particle after the measurement?

Problem 3: Matrix representation of operator

Operators are very common in linear algebra. We will use linear algebra a lot to calculate operators in the second half of this course. Operator can be considered as a gate which acts on the information carried by quantum states (wave function). Now consider a system with two eigenstates (you can imagine they are the two lowest energy states of a quantum well). If we only measure the energy of the state (not position of the electron), then we don't really care about the form of the wave function. We can simply use two orthonormal vectors to represent the states.

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3}$$

(a) Show that the dot product of vector ψ_1 and ψ_2 is zero.

(b) Show that the dot product of vector ψ_1 and ψ_1 is 1.

(c) Show that the dot product of vector ψ_2 and ψ_2 is 1.

(d) Based on your answers of (a)-(c), do you think dot product of two real vectors is similar to integration of two wave functions $\int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)dx$?

Operators that operate on ψ_1 and ψ_2 can be expressed by a two-by-two matrix:

$$\hat{O} = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}. \tag{4}$$

(e) Identity operator, $\hat{I}$: an operator that leaves unchanged the element on which it operates. For example: $\hat{I}\psi_1 = \psi_1$ and $\hat{I}\psi_2 = \psi_2$. What are the numerical values of I_{11}, I_{12}, I_{21} and I_{22}?

(f) Operator $\hat{\sigma^z}$ can change state ψ_1 to state ψ_2, and state ψ_2 to state ψ_1, such that $\hat{\sigma^z}\psi_1 = \psi_2$ and $\hat{\sigma^z}\psi_2 = \psi_1$. What are the numerical values of σ^z_{11}, σ^z_{12}, σ^z_{21} and σ^z_{22}? This is the famous Pauli-X gate.