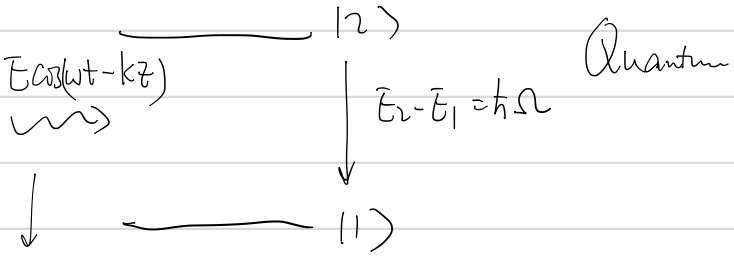
Week 7 and 8.

Quantum aspect of light.

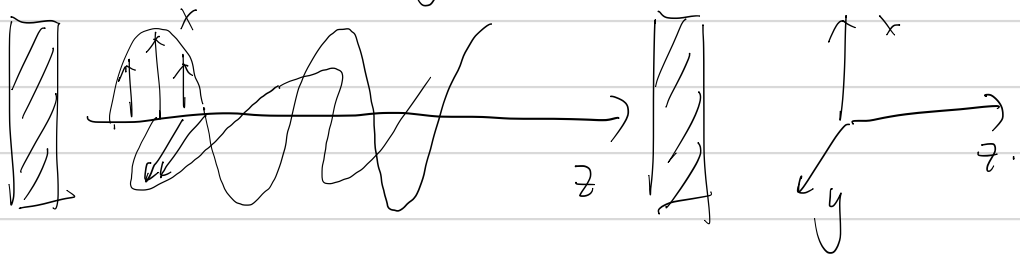
We've talked a lot of one photon has energy of $\hbar\omega$, but we've never shown how we derived it.

In addition, there's an extreme case in light-matter interactin and it is related to quantum computer/information: One photon interact with One atom.



Classic: This is the part we want to quantize.

To simplify our calculation, we restrict ourselves to light in a FP cavity (two mirrors)



Electric field: $\vec{E}$: $\vec{x}$ directin
Magnetic field: $\vec{H}$: $\vec{y}$ directin
Light propagation directin : $\vec{z}$.

Because it's a cavity with two mirror, we know:

$\vec{E}$, $\vec{H}$ oscillate periodically (mode distribution)

spatially

$\therefore \vec{E}(t,z) = \vec{E}(t)\sin kz$; because it's a cavity, we have periodic condition: $j\lambda = 2L$; L is the length of the cavity, j is an integer number.

$$\therefore k_j = \frac{2\pi}{\lambda_j} = \frac{2\pi j}{2L} = \frac{\pi j}{L}.$$

If we want to calculate light in free-space, let $L \to \infty$.

To do quantum mechanics, we have to write down the Hamiltonian of the light.

Hamiltonian is energy, so:

$$H = \frac{1}{2}\int (\epsilon_0 E^2 + \mu_0 H^2)\, dV$$

Let's have: $\vec{E}(t,z) = A\, q(t)\cdot \sin kz \cdot \vec{e}_x$

This is something you can do in a cavity: seperate time and space dependent.

$$\therefore \frac{1}{2}\int \varepsilon_0 \vec{E}^2 dV = \frac{1}{2}\int \varepsilon_0 A^2 q^2(t) \sin^2 kz \cdot dz \cdot dx dy$$

We want it to look like something we are familiar with:

Harmonic oscillator:

$$H_{HO} = \frac{1}{2}kx^2 + \frac{p^2}{2m} \; ; \quad \omega^2 = \frac{k}{m} \; ; \quad p = mv = m \cdot \frac{dx}{dt}.$$

$$= \frac{1}{2} m\omega^2 x^2 + \frac{p^2}{2m} \; ;$$

Is $q^2(t)$ similar to x^2, or p^2 ?

Decide by $\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$; So $\vec{H}$ vs $\dot{q}(t) = \frac{dq(t)}{dt}$.

So $\vec{H}$ term will go to p^2, $\vec{E}$ term $\rightarrow x^2$.

So: $\frac{1}{2}\int \varepsilon_0 A^2 q^2(t) \sin^2 kz \, dx dy dz = \frac{1}{2} S_{area} \cdot \frac{L}{2} \cdot \varepsilon_0 A^2 \cdot q^2(t)$

And $S_{area} \cdot L = V$ volume of the cavity

$$\therefore \frac{1}{2}\int \varepsilon_0 E^2 dV = \frac{1}{2} \cdot \frac{\varepsilon_0 V}{2} \cdot A^2 q^2(t) \equiv \frac{1}{2} m\omega^2 q^2(t)$$

$$\therefore A^2 = \frac{2m\omega^2}{\varepsilon_0 V} \Rightarrow A = \sqrt{\frac{2m\omega^2}{\varepsilon_0 V}}.$$

$$\therefore \vec{E} = \vec{e_x} \cdot \sqrt{\frac{2m\omega^2}{\varepsilon_0 V}} \, q(t) \sin kz$$

Now calculate $\vec{H}$. $\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\Rightarrow \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = -\vec{e}_x \cdot \frac{\partial}{\partial z} H_y = \vec{e}_x \sqrt{\frac{2m\omega^2}{\varepsilon_0 V}} \cdot \dot{q}(t) \sin kz \cdot \varepsilon_0$$

Easy solution is: $H_y = \frac{1}{k} \cos kz \cdot \sqrt{\frac{2m\omega^2}{\varepsilon_0 V}} \cdot \dot{q}(t) \cdot \vec{e}_y \cdot \varepsilon_0$

$$\Rightarrow \vec{H} = \vec{e}_y \cdot \frac{\varepsilon_0}{k} \cdot \sqrt{\frac{2m\omega^2}{\varepsilon_0 V}} \cdot \dot{q}(t) \cdot \cos kz$$

$$\therefore \frac{1}{2}\mu_0 \int \vec{H}^2 dV = \frac{\mu_0 \varepsilon_0}{2} \cdot \frac{1}{k^2} \cdot \frac{2m\omega^2}{V} \cdot \int \underbrace{\cos^2 kz \cdot dx dy dz}_{= \frac{1}{2}V} \cdot \dot{q}^2(t)$$

$$= \frac{1}{2} \cdot m \frac{\mu_0 \varepsilon_0 \cdot \omega^2}{k^2} \cdot \dot{q}^2(t) \; ; \quad \text{we know: } k = \frac{2\pi}{\lambda} = \frac{2\pi}{c \cdot T} = \frac{2\pi f}{c} = \frac{\omega}{c}$$

Notice: $\mu_0 \varepsilon_0 = \frac{1}{c^2}$; $\therefore$

$$\frac{1}{2}\mu_0 \int \vec{H}^2 dV = \frac{1}{2} m \dot{q}^2(t) \; ;$$

In total: $H = \frac{1}{2}\int \varepsilon_0 E^2 + \mu_0 H^2 dV = \frac{m\omega^2}{2} q^2(t) + \frac{m}{2}\dot{q}^2(t)$

$$= \frac{1}{2} m\omega^2 q^2(t) + \frac{p^2(t)}{2m} \; ; \quad p(t) = m \cdot \dot{q}(t) \; .$$

Same as Harmonic oscillator.

So the solutions for harmonic oscillator all work for photon.

Cross-check: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$; If you plug in $\vec{E}$ and $\vec{B}$,

we will get: $-m\omega^2 q(t) = m \cdot \ddot{q}(t)$

$-m\omega^2 q(t)$: force of a spring; $m \cdot \ddot{q}(t)$: ma.

Up to this point, no quantum mechanics.
We simply draw connection between light in cavity and harmonic oscillator.

Quantum part: For harmonic oscillator, we can solve its eigenstate and eigen-energy, it is:

$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$; $H \cdot \psi_n = E_n \cdot \psi_n \rightarrow$ Solve this equation.

The procedure is very very complicated and we skip it.

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right);$$

$$\psi_n = \frac{1}{\sqrt{2^n \cdot n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega q^2}{2\hbar}} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, q\right) ; \quad n = 0, 1, 2, 3, \ldots$$

H_n : Hermite polynomials:

$$H_n(z) = (-1)^n e^{z^n} \frac{d^n}{dz^n}\left(e^{-z^2}\right)$$

We move further:

Creation and annihilation operator.

Question: is there an operator $\hat{O}$, that add a photon to the system? $\hat{O}\psi_n \to \psi_{n+1}$.

If so, the emission and absorption process becames very easy to express: Emission: $\hat{O}\cdot|\psi\rangle$

Absorption: $\hat{O}^+|\psi\rangle$.

And the simplest guess of the operator is linear combinition of $q(t)$ and $p(t)$.

The second arguement is that $\hat{O}$'s dimension is 1

$$\hat{O}|\psi_n\rangle \to A_n\cdot|\psi_{n-1}\rangle$$

A_n: Number.

for simplicity, $q(t)\to x(t)$; $p(t)\to p(t) = m\frac{dx}{dt}$.

$$\beta_x x(t) + \beta_p p(t)$$

$$\hbar\omega \sim \frac{1}{2}m\omega^2x^2 \quad ; \quad x \sim \sqrt{\frac{2\hbar}{m\omega}} \qquad \beta_x \propto \sqrt{\frac{m\omega}{2\hbar}}$$

Similarly: $\hbar\omega \sim \frac{p^2}{2m}$; $p \sim \sqrt{2\hbar\omega m}$ $\therefore \beta_p \propto \sqrt{\frac{1}{2\hbar m\omega}}$

$\hat{a}$ is linear combinition of $\left(\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \qquad \sqrt{\frac{1}{2\hbar m\omega}}\,\hat{p} \right)$.

Combinition can be: $\sim \hat{x} \pm \hat{p}$

$\sim \hat{x} \pm i\hat{p}$.

Which one to choose? We know: $\hat{a}|\psi_n\rangle \to |\psi_{n-1}\rangle$

$\hat{a}^+|\psi_n\rangle \to |\psi_{n+1}\rangle$

Reverse process. "+" Hermitian conjugate

Since $\hat{x}$ and $\hat{p}$ are all real measureable quantity, they are Hermit; $\hat{x}^+ = \hat{x}$; $\hat{p}^+ = \hat{p}$.

So to make $\hat{a} \neq \hat{a}^+$, one has to choose i in the linear combinitim:

$$\hat{a} = \left(\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\,\hat{p} \right)$$

$$\hat{a}^+ = \left(\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - i\sqrt{\frac{1}{2\hbar m\omega}}\,\hat{p} \right)$$

Now the only other thing we know is

$$H|\psi_n\rangle = E_n|\psi_n\rangle; \quad H = \frac{1}{2}m\omega^2x^2 + \frac{p^2}{2m}; \quad E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

So plug a and $\hat{a}^\dagger$ back to H:

$$\hat{a} + \hat{a}^\dagger = 2\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^\dagger)$$

$$\hat{a} - \hat{a}^\dagger = 2i\sqrt{\frac{1}{2\hbar m\omega}}\,\hat{p} \Rightarrow \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^\dagger - \hat{a})$$

plug into H: $H = \frac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{2m}$

$$= \frac{1}{4}\hbar\omega\,(\hat{a} + \hat{a}^\dagger)^2 - \frac{\hbar\omega}{4}\,(\hat{a}^\dagger - \hat{a})^2$$

$$= \frac{\hbar\omega}{4}\left[\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} - \hat{a}^{\dagger 2} - \hat{a}^2 + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right]$$

$$= \frac{\hbar\omega}{2}\left[a^\dagger a + aa^\dagger\right]$$

The only thing missing, is $[a, a^\dagger] = ?$ $[\ ]$: Commutator.

$[a, a^\dagger] = aa^\dagger - a^\dagger a$. In classical physics $\longrightarrow 0$.
because $xp = px$.

But in Quantum mechanics, it's different.

$\hat{p} = \hbar\vec{k}$; what is $\vec{k}$? $\frac{\partial}{\partial x}e^{ikx} = kx$.

$$\hat{p} = \frac{\hbar\frac{\partial}{\partial x}}{i} = -i\hbar\frac{\partial}{\partial x}.$$

$$\hat{x}\hat{p} = -i\hbar \cdot x\frac{\partial}{\partial x}; \quad \hat{p}\cdot\hat{x} = -i\hbar\frac{\partial}{\partial x}\cdot x$$

$$\hat{p}\hat{x}\psi(x) = -i\hbar\frac{\partial}{\partial x}(x\psi(x)) = -i\hbar\cdot\psi(x) - i\hbar x\cdot\frac{\partial}{\partial x}\psi(x)$$

$$= (-i\hbar + \hat{x}\hat{p})\,\psi(x).$$

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar = [\hat{x}, \hat{p}]$$

And because $\hat{a}$, $\hat{a}^+$ are linear combinition of $\hat{x}$, $\hat{p}$, so cannot switch; $\hat{a}\hat{a}^+ \neq \hat{a}^+\hat{a}$.

$$[\hat{a}, \hat{a}^+] = \left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}\right)$$

$$- \left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}\right)$$

$$= \frac{m\omega}{2\hbar}\hat{x}^2 - \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{\hat{p}^2}{2\hbar m\omega} - \frac{\hat{p}^2}{2\hbar m\omega}$$

$$\sqrt{\frac{m\omega}{2\hbar}\cdot\frac{}{2\hbar m\omega}}\left\{-i\hat{x}\hat{p} + i\hat{p}\cdot\hat{x} - i\hat{x}\hat{p} + i\hat{p}\hat{x}\right\}$$

$$= \frac{-i}{\hbar}\left\{\hat{x}\hat{p} - \hat{p}\hat{x}\right\} = \frac{-i}{\hbar}[\hat{x}, \hat{p}] = \frac{-i}{\hbar}\cdot i\hbar = 1$$

$$\hat{a}\hat{a}^+ = \hat{a}^+\hat{a} + 1$$

plug this back to H:

$$H = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

As we know: $H\psi_n = \hbar\omega\left(n+\frac{1}{2}\right)\psi_n$

$$\therefore \hat{a}^\dagger\hat{a}\,\psi_n = n\psi_n ;$$

$N = \hat{a}^\dagger\hat{a}$: number operator.

This is very cool. For any arbitary state $|\psi\rangle$, $\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle$ gives the average photon number.

Now let's some arguement on what $\hat{a}$ and $\hat{a}^\dagger$ do to ψ_n.

If $\hat{a}|\psi_n\rangle$ lower the number of photon, then it must lower by 1. Otherwise:

$$\hat{a}|\psi_1\rangle \to |\psi_{-1}\rangle \text{ , non-exist, } = 0.$$

$$\therefore \hat{a}^\dagger\hat{a}|\psi_1\rangle = 0 \neq 1\cdot|\psi_1\rangle \text{ ; Contradiction.}$$

If $\hat{a}|\psi_n\rangle$ increase the number of photon,
then: $\hat{a}|\psi_0\rangle \to |\psi_1\rangle$; $\hat{a}^\dagger|\psi_1\rangle \to |\psi_0\rangle$.
$\hat{a}^\dagger\hat{a}|\psi_0\rangle \neq 0$.

So: $\hat{a}|\psi_n\rangle \to |\psi_{n-1}\rangle$; $\hat{a}^\dagger|\psi_n\rangle \to |\psi_{n+1}\rangle$.

Another way to verify:

$$\hat{a} = \left(\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2\hbar m\omega}}\,\hat{p}\right); \quad \hat{p} = -i\hbar\frac{\partial}{\partial x};$$

$$\psi_n = \frac{1}{\sqrt{2^n \cdot n!}}\left(\frac{m\omega}{2\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right); \quad n = 0, 1, 2, 3, \ldots$$

H_n : Hermite polynomials:

$$H_n(z) = (-1)^n e^{z^n}\frac{d^n}{dz^n}\left(e^{-z^2}\right)$$

Calculate $\hat{a}\psi_n$, and you will see $\hat{a}\psi_n \to \psi_{n-1}$

Now determine the factor:

$$\langle n|a^\dagger a|n\rangle = \langle n|n|n\rangle = n$$

$$\langle n|a^\dagger = (a|n\rangle)^\dagger \quad ; \quad \text{Let: } a|n\rangle = A_n|n\rangle \; ; \; (a|n\rangle)^\dagger = A_n^*\langle n|$$

$$\therefore \langle n|a^\dagger = A_n^*\langle n|$$

$$\therefore \langle n|a^\dagger a|n\rangle = A_n \cdot A_n^*\langle n|n\rangle = |A_n|^2 = n$$

$A_n = \sqrt{n}$. (choose phase zero, simplicity)

$$a|n\rangle = \sqrt{n}\,|n-1\rangle;$$

Now $a^\dagger a|n\rangle = n|n\rangle = a^\dagger \cdot \sqrt{n}|n-1\rangle = \sqrt{n}\, a^\dagger|n-1\rangle$

$\therefore a^\dagger|n-1\rangle = \sqrt{n}\,|n\rangle$

$\therefore a^\dagger|n\rangle = \sqrt{n+1}\,|n\rangle$.

In conclusion, we have:

$$H = \frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{p} \quad ; \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}$$

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \quad ; \quad [a, a^\dagger] = aa^\dagger - a^\dagger a = 1$$

$$a|n\rangle = \sqrt{n}\,|n\rangle ; \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n\rangle.$$

Now back to $\vec{E}$ field:

$$\vec{E} = \vec{e_x}\cdot\sqrt{\frac{2m\omega^2}{\varepsilon_0 V}}\, q(t)\sin kz \quad ; \text{ put back } q(t) = x(t)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$$

$$\vec{E} = \hat{e}_x\sqrt{\frac{2m\omega^2}{\varepsilon_0 V}}\cdot\sqrt{\frac{\hbar}{2m\omega}}\cdot\left(\hat{a} + \hat{a}^\dagger\right)\sin kz$$

$$= \hat{e}_x\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\cdot\left(\hat{a} + \hat{a}^\dagger\right)\sin kz$$

We actually neglect $e^{-i\omega t}$ term in all these calculation, as I want to make it simple enough for one class.

$$\vec{E} = \hat{e}_x \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} + \hat{a}^+ e^{+i\omega t} \right) \sin kz .$$

Further Extend to free-space:

$$\vec{E} = \vec{e} \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t + i\vec{k}\cdot\vec{r}} + \hat{a}^+ e^{i\omega t - i\vec{k}\cdot\vec{r}} \right)$$

$\mathcal{E} \equiv \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}$: The electric field amplitude of 1-photon.

If there's more than one frequency component:

$$\vec{E} = \sum_j \vec{e}_j \, \mathcal{E}_j \left(\hat{a}_j e^{-i\omega_j t + i\vec{k}_j\cdot\vec{r}} + \hat{a}_j^+ e^{i\omega_j t - i\vec{k}_j\cdot\vec{r}} \right) \qquad [a_j, a_k^+] = \delta_{j,k}$$

This is the quantized electric field of light.

Use this to replace $\vec{E}$ whenever you want to calculate things in Quantum regime